COMPLEX STOCHASTIC NOISE AND MODULATION

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Abstract

Distribution theory is utilized to represent stochastic modulation and noise processes as analytic time functions. These representations are used to obtain new results in single sideband angle demodulation in the presence of noise.

Introduction

The usefulness of the representation of modulated signals by analytic time functions has been well established by Gabor,(1) Oswald,(2) Bedrosian,(3) and others.[1] The representation of stochastic, or "noise," processes by analytic functions has received less attention, being treated primarily by Middleton,(4) Dugundji,(5) Zakai,(6) and Belyaev.(7)

Representation of modulation and noise by analytic time functions is made intuitionally and mathematically acceptable through use of the powerful tools of distribution theory. Works such as those of Gel'Fand and Shilov,(8) Zemanian,(9) and Jones,(10) derive the few properties of distributions which are needed to provide an enlightening, yet simple, method for treating stochastic modulation and noise processes.

A systematic development of the theory is applied in this paper to obtain new results in demodulation of SSB angle-modulated signals in the presence of noise.

An Analytic Distributional Convolution Transform

Deterministic Functions

Let us postulate a nonphysical linear system, shown in Figure 1, whose input is a real time function, x(t), and whose output is the analytic form of x(t) denoted z(t). That is, z(t) satisfies the Cauchy-Riemann equations on the real t-axis of the complex plane, and

$$z(t) = x(t) + jy(t) \tag{1}$$

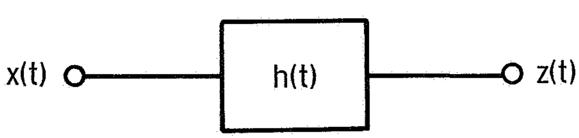

Figure 1.- Nonphysical system.

The impulse response h(t), of such a system is a complex singular distribution given by[2]

$$h(t) = \delta(t) + j\,\frac{1}{\pi t} \tag{2}$$

This may be shown by writing, formally, the convolution integral relating x(t), h(t), and z(t).

$$z(t) = \int_{-\infty}^{\infty} x(\sigma)\left[\delta(t-\sigma) + j\,\frac{1}{\pi(t-\sigma)}\right]d\sigma$$

$$= x(t) + j\,\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\sigma)}{t-\sigma}\,d\sigma \tag{3}$$

It should be understood that the convolution integral in equation (3) exists only in the distributional sense and is given as a conceptual device only. The integral in the imaginary term of equation (3) is readily identified as the Hilbert transform of x(t). Hence, according to Titchmarsh,(13) z(t) is analytic whenever the Hilbert transform of x(t) exists.

We may interpret h(t) as a "kernel" for a linear, convolution-type transform from real to analytic time functions. The real function can always be recovered simply by taking the real part of the analytic function.

Stochastic Processes

When x(t) is deterministic, z(t) is deterministic. When x(t) is stochastic, z(t) is stochastic. The existence of z(t) in the deterministic case may be inferred from the existence of the Hilbert transform of x(t). For the stochastic case, it is simpler to determine existence of z(t) from conditions on the covariance given by Belyaev.(7)

[1]See Bedrosian(3) and Voelcker(11) for bibliography.

[2]Kailath(12) has used this distribution in a related application.

Let us derive the covariance relations for our model of Figure 1, under the mild restrictions that $x(t)$ is stationary in the wide-sense and of zero mean value. Then the covariance and autocorrelation function of $x(t)$ are identical. Also, the autocorrelation function, $R_{xx}(\tau)$, is time-invariant, varying only with time displacement, τ. Hence, the autocorrelation function is deterministic.

Because of linearity, $z(t)$ is weakly stationary and the autocorrelation function is

$$\begin{aligned} R_{zz}(\tau) &\triangleq E\left\{z(t+\tau)z^*(t)\right\} \\ &= R_{xx}(\tau) * h(\tau) * h^*(-\tau) \end{aligned} \tag{4}$$

where E denotes the statistical expectation, the midline asterisk denotes convolution integral, and the superscript asterisk denotes complex conjugate. It may be determined that

$$h(\tau) * h^*(-\tau) = 2h(\tau) \tag{5}$$

Thus

$$\begin{aligned} R_{zz}(\tau) &= R_{xx}(\tau) * 2h(\tau) \\ &= 2\left[R_{xx}(\tau) + j\hat{R}_{xx}(\tau)\right] \end{aligned} \tag{6}$$

where the caret denotes Hilbert transform. Since $R_{xx}(\tau)$ is deterministic, $R_{zz}(\tau)$ is analytic whenever the Hilbert transform of $R_{xx}(\tau)$ exists. Now, Belyaev(7) shows that a sufficient condition for the existence of an analytic stochastic process is the analyticity of its covariance. Thus, we see that the process, $z(t)$, at the output of our nonphysical linear system, is analytic provided the autocorrelation of the real function, $x(t)$, has a Hilbert transform.

The cross-correlation functions between $x(t)$ and $z(t)$ may be developed as was the autocorrelation function to obtain

$$R_{zx}(\tau) = R_{xz}(\tau) = \frac{1}{2} R_{zz}(\tau) \tag{7}$$

Then using the analyticity of $z(t)$, equations (6) and (7), and the general relation

$$R^*_{zz}(-\tau) = R_{zz}(\tau) \tag{8}$$

we obtain

$$\left.\begin{aligned} R_{xx}(\tau) &= R_{yy}(\tau) \\ R_{yx}(\tau) &= -R_{xy}(\tau) = -R_{yx}(-\tau) = \hat{R}_{xx}(\tau) \end{aligned}\right\} \tag{9}$$

Similar relations were obtained by Dugundji(5) using time averages. His results were extended to weakly stationary processes by Zakai,(6) using a frequency domain approach to the Hilbert transform.

The power spectral density of $z(t)$, denoted $S_{zz}(\omega)$ may be developed from equation (6) as the Fourier transform of $R_{zz}(\tau)$

$$S_{zz}(\omega) = 2S_{xx}(\omega)H(\omega) \tag{10}$$

where $H(\omega)$ and $S_{xx}(\omega)$ are the Fourier transforms of the impulse response, $h(\tau)$, and of the autocorrelation of $x(t)$, respectively. $H(\omega)$ exists only in a distributional sense,(9) and is given by

$$H(\omega) = 1 + \operatorname{sgn}(\omega) = \left\{\begin{array}{ll} 2 ; & \omega > 0 \\ 1 ; & \omega = 0 \\ 0 ; & \omega < 0 \end{array}\right\} \tag{11}$$

thus

$$S_{zz}(\omega) = \left\{\begin{array}{ll} 4\, S_{xx}(\omega) ; & \omega > 0 \\ 2\, S_{xx}(\omega) ; & \omega = 0 \\ 0 ; & \omega < 0 \end{array}\right\} \tag{12}$$

Equation (12) shows that forming an analytic process or function from its real part quadruples the positive frequency components and deletes the negative frequency components.

By a development similar to the above it may be shown that convolving the conjugate analytic kernel, $h^*(t)$, with a real function produces a conjugate analytic function, $z^*(t)$. The autocorrelation for this function is also conjugate analytic. The power spectral density exists only for negative frequencies.

Bandpass Processes

Bandpass processes may be defined rather generally as processes with spectral densities which have a finite frequency support not extending to the origin. Such processes are usually referred to particular frequency ω_c.

However, the spectral density need not be symmetric over the positive frequencies with respect to ω_c or any other frequency. Using complex time function notation, both signal processes and noise processes may be treated in essentially the same manner[3].

[3]Helstrom(14) has made a similar observation.

where $\phi(t)$ is analytic. For a deterministic f(t), we have

$$g(t) = \hat{f}(t) \tag{25}$$

and

$$A(t) = A \exp\left[-\hat{f}(t)\right] \tag{26}$$

Suppose $\phi(t)$ is a zero-mean, weakly stationary complex Gaussian process. Then f(t) and g(t) are individually and jointly Gaussian and weakly stationary. The autocorrelation function of the analytic signal, $\psi(t)$, is

$$R_{\psi\psi}(t + \tau, t) = A^2 \exp(j\omega_c \tau) E\left\{\exp\left[jf(t + \tau) - g(t + \tau) - jf(t) - g(t)\right]\right\} \tag{27}$$

The expectation in equation (27) has the form of a characteristic function. The autocorrelation function reduces to

$$R_{\psi\psi}(\tau) = A^2 \exp\left[j\omega_c \tau\right] \exp\left\{2\left[R_{ff}(\tau) + jR_{gf}(\tau)\right]\right\} \tag{28}$$

For $\psi(t)$ to represent an upper sideband signal, we must have

$$R_{gf}(\tau) = \hat{R}_{ff}(\tau) \tag{29}$$

and, thus, $\phi(t)$ is analytic, as supposed.

Detection Results

Let us consider the linear product demodulation of a single-sideband angle modulated carrier in the presence of noise. A deterministic sinusoidal modulating signal is chosen. Figure 2 shows the model. Optimum bandpass filtering is employed at the input which ideally passes the signal, undistorted, and only the noise above ω_c. $n_i(t)$ is band-limited white Gaussian noise of the form of equation (17) where $x(t) + jy(t)$ is analytic. $s_i(t)$ and $s_r(t)$ are

$$\left.\begin{aligned} s_i(t) &= A \exp\left[-\beta \sin \omega_m t\right] \cos\left[\omega_c t + \beta \cos \omega_m t\right] \\ s_r(t) &= -2 \sin \omega_c t \end{aligned}\right\} \tag{30}$$

Then, the ratio of output signal (modulation) to noise, $\left.\frac{S_o}{N_o}\right]_{B_o}$, is related to the ratio of input signal (total) to noise, $\left.\frac{S_i}{N_i}\right]_{B_i}$, as

$$\left.\frac{S_o}{N_o}\right]_{B_o} = \left[\frac{B_i}{B_o}\right]\left[\frac{\beta^2}{I_o(2\beta)}\right]\left.\frac{S_i}{N_i}\right]_{B_i} \tag{31}$$

where I_o is the modified Bessel function. The "modulation gain factor," the function of β in brackets in equation (31), has a maximum value of 0.475 for $\beta = 1.29$. Hence, this type of modulation system is inherently a "low-index" system.

Next, consider the limiter-discriminator demodulation of a single-sideband angle modulated carrier in the presence of noise. A Gaussian modulating process is chosen. Figure 3 shows the model where, again, optimum, ideal, input filtering is assumed. The noise process $n_i(t)$ into the limiter is the same as for the previous example. The input signal is

$$s_i(t) = A \exp\left[-g(t)\right] \cos\left[\omega_c t + f(t) + \theta\right] \tag{32}$$

where θ is a uniformly distributed random constant and $f(t) + jg(t)$ is an analytic Gaussian process. A lowpass spectral density is assumed for f(t) as

$$S_{ff}(\omega) = K \exp\left[-\left(\frac{\omega}{\omega_o}\right)^2\right] \tag{33}$$

where ω_o is some upper cut-off frequency and K is a constant.

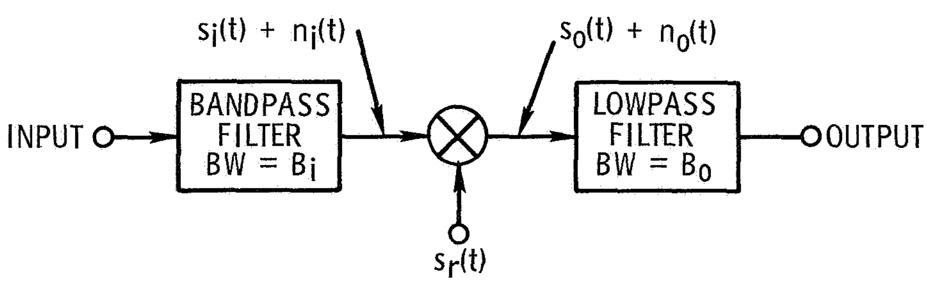

Figure 2.- Linear demodulator model.

Noise Processes

Let us now generalize our process, $z(t)$, so that it is not necessarily analytic. An analytic bandpass process, $\nu(t)$, may be defined as

$$\nu(t) = z(t)\exp\left[j\omega_c t\right] \tag{13}$$

for which ω_c is simply some reference frequency by which the spectrum of $z(t)$ is translated. Under the restriction that $x(t)$ and $y(t)$ are individually and jointly stationary in the wide sense, $z(t)$ is weakly stationary and $\nu(t)$ has autocorrelation function

$$\begin{aligned} R_{\nu\nu}(\tau) &= R_{zz}(\tau)\exp\left[j\omega_c\tau\right] \\ &= \Big\{R_{xx}(\tau) + R_{yy}(\tau) \\ &\quad + j\left[R_{yx}(\tau) - R_{xy}(\tau)\right]\Big\}\exp\left[j\omega_c\tau\right] \end{aligned} \tag{14}$$

The spectral density of $\nu(t)$ is then

$$S_{\nu\nu}(\omega) = S_{zz}(\omega - \omega_c) \tag{15}$$

For $\nu(t)$ to be analytic requires that $S_{\nu\nu}(\omega)$ exist only for positive frequency. Hence, a lower bound is placed on the spectrum of $z(t)$ as

$$S_{zz}(\omega) = 0 \; ; \; \omega < -\omega_c \; ; \; \nu(t)\text{analytic} \tag{16}$$

The result of equation (16) has been stated by Nuttal.[15]

The physical process associated with the analytic process, $\nu(t)$, is defined as

$$n(t) = \mathrm{Re}\left\{\nu(t)\right\} = x(t)\cos\omega_c t - y(t)\sin\omega_c t \tag{17}$$

Now, it is required that $n(t)$ be weakly stationary. Whereas the weak stationarity of $\nu(t)$ required only that $z(t)$ be weakly stationary, the weak stationarity of $n(t)$ requires also that

$$\left.\begin{aligned} R_{yy}(\tau) &= R_{xx}(\tau) \\ R_{yx}(\tau) &= -R_{xy}(\tau) = -R_{yx}(-\tau) \end{aligned}\right\} \tag{18}$$

Equations (9) obviously satisfy equations (18). Hence, a sufficient condition for a weakly stationary, zero-mean, bandpass stochastic process to be represented in the familiar form[16] of equation (17) is that $z(t)$ be analytic. This means that the spectrum of $z(t)$ exists only for positive frequency. Hence, the spectra of $\nu(t)$ and $n(t)$ lie entirely above the reference frequency, ω_c. Similar reasoning shows that it is also permissible for ω_c to be above the spectrum of $n(t)$. For the case where ω_c lies within the spectrum of $n(t)$, equations (18) are satisfied if $z(t)$ is complex Gaussian.[17] Then $x(t)$ and $y(t)$ are of zero mean and individually and jointly weakly stationary and Gaussian.

For either case, the autocorrelation functions are given as

$$R_{\nu\nu}(\tau) = 2\left[R_{xx}(\tau) + jR_{yx}(\tau)\right]\exp\left[j\omega_c\tau\right] \tag{19}$$

$$\begin{aligned} R_{nn}(\tau) &= \frac{1}{2}\,\mathrm{Re}\left\{R_{\nu\nu}(\tau)\right\} \\ &= R_{xx}(\tau)\cos\omega_c\tau - R_{yx}(\tau)\sin\omega_c\tau \end{aligned} \tag{20}$$

Signal Processes

Let us generalize $z(t)$ further as

$$z(t) \triangleq A(t)\exp\left[jf(t)\right] \tag{21}$$

where $A(t)$ is a real "amplitude function" and $f(t)$ is a real "phase function." We may now define an "analytic signal"[3] as

$$\psi(t) = z(t)\exp(j\omega_c t) \tag{22}$$

The real part of $\psi(t)$, denoted $s(t)$, is a "modulated carrier" in its most general form.

$$s(t) = \mathrm{Re}\left\{\psi(t)\right\} = A(t)\cos\left[\omega_c t + f(t)\right] \tag{23}$$

Various classical forms of modulation (FM, AM) may be recognized by setting $A(t)$ or $f(t)$ equal to constants.

Single-Sideband Angle Modulation Examples

Signal Properties

Consider now a single-sideband angle-modulated carrier. Let a deterministic $f(t)$, or its derivative, represent the transmitted message. $z(t)$ must be analytic, since, then, the spectrum of $s(t)$ will lie entirely above the carrier frequency, ω_c. Then

$$\left.\begin{aligned} z(t) &= A\exp\left[j\phi(t)\right] \\ \phi(t) &= f(t) + jg(t) \end{aligned}\right\} \tag{24}$$

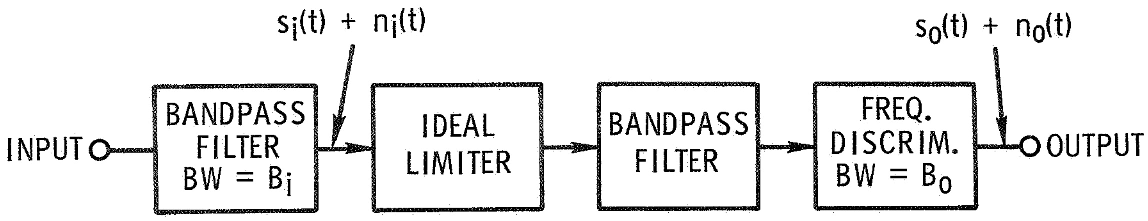

Figure 3.- Nonlinear demodulator model.

Now, the system of Figure 3 is an FM demodulator. The chief purpose of FM is to obtain "modulation gain factors" greater than unity by use of high modulation indices. Hence, we are interested in the relation of output to input signal-to-noise ratios (SNR) under the assumption that the modulation index is high. Since we are interested in the "best" operation of this device, we also make the assumption that the input SNR is high. Under these assumptions the input-output SNR relation is obtained, approximately, as

$$\left.\frac{S_o}{N_o}\right]_{B_o} \cong 2\left[\frac{B_i}{B_o}\right]\left[\exp\left(-4\sigma_f^2\right)\right]\left.\frac{S_i}{N_i}\right]_{B_i} \tag{34}$$

where σ_f^2 is the variance of $f(t)$. σ_f is the r.m.s. modulation index. Thus, we see that under the assumptions, high-index single-sideband frequency modulation produces, not modulation gain, but, rather, modulation loss.

Conclusion

This paper has attempted to provide a simple theoretical framework for treating both signal and noise processes in a way which makes best use of properties which they have in common. Existence of the analytic counterpart of a real deterministic time function has been related to existence of the Hilbert transform of the real function. Existence of the analytic counterpart of a real stochastic process has been related to existence of the Hilbert transform of the autocorrelation function of the real process. New results in demodulation of single-sideband angle modulated signals in the presence of noise have been obtained through application of the theory.

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